Self–gravitating cosmic rings

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Abstract

The classical Einstein–Maxwell field equations admit static horizonless wormhole solutions with only a circular cosmic string singularity. We show how to extend these static solutions to exact rotating asymptotically flat solutions. For a suitable range of parameter values, these solutions describe charged or neutral rotating closed cosmic strings, with a perimeter of the order of their Schwarzschild radius.

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The Einstein–Maxwell field equations coupling gravity to electromagnetism admit outside sources a variety of stationary axially symmetric solutions, among which the Kerr–Newman family of solutions [1] depending on three parameters M, Q and J which, from the consideration of the asymptotic behaviour of these field configurations, may be identified as their total mass, charge, and angular momentum. A fourth physical characteristic, the total magnetic moment μ, is related to these three parameters by the same "anomalous" relation

$$g \equiv 2\frac{M\mu}{JQ} = 2 \tag{1}$$

as in the case of an elementary particle such as the electron. It would be tempting to interpret these classical configurations as the fields generated by an isolated elementary particle, were it not for the numerical values of these parameters. The Kerr–Newman metrics correspond to regular black–hole spacetimes if

$$M^2 \geq Q^2 + a^2, \tag{2}$$

where[1] $a \equiv J/M$. However in the case of elementary particles $|J| \sim m_P^2$ (where $m_P = (\hbar c/G)^{1/2}$ is the Planck mass) and $|Q| \sim m_P$, so that

$$|a|/|Q| \sim |Q|/M \sim m_P/m_e \sim 10^{22}$$

(where m_e is the electron mass), and therefore

$$M^2 < Q^2 + a^2\,. \tag{3}$$

So the field configurations generated by elementary particles are not of the black hole type, but exhibit naked singularities, violating the cosmic censorship hypothesis [2]. For this reason, it is generally believed that there is no viable classical model for elementary particles (except possibly for neutral spinless particles) in the framework of Einstein's general relativity.

In the charged spinless case $J = 0$, the relation $|Q|/M \sim m_P/m_e$ tells us that electromagnetism is preponderant and that the point singularity in the spherically symmetric metric originates from that of the Coulomb central field. In classical models of (spinless) charged point particles, this singularity leads to the divergence of the particle's self–energy, and the situation becomes much worse in quantum field theory. These divergences may be tamed in quantum string theories where the elementary objects are no longer zero–dimensional (point particles) but one–dimensional (strings). Similarly, one expects that the divergences in a classical field theory with line sources will be less severe than with point sources. String–like objects occur as classical solutions to field theories with spontaneously broken global or local symmetries. Such symmetry–breaking transitions are believed to have occurred during the expansion of the universe, leading to the formation of large, approximately straight cosmic strings [3]. The long–range behavior of the metric generated by a straight cosmic string is given by an exact stationary solution of the vacuum Einstein equations with a line source carrying equal mass per unit length and tension [4]. In the case of closed cosmic strings, or rings, this tension will cause the string to contract, precluding the existence of stationary solutions, unless the tension is balanced by other forces (superconducting cosmic strings [5], vortons [6]). Another

[1] We use gravitational units $G = 1$, $c = 1$.

possibility has been advocated by Bronnikov and co–workers [7], that of ring wormhole solutions to multi–dimensional field models. In the case of these static solutions, the gauge field energy–momentum curves space negatively to produce a wormhole, at the neck of which sits a closed cosmic string, which cannot contract because its circumference is already minimized.

In this Letter, we first rederive such static ring wormhole solutions to the Einstein–Maxwell field equations. Then, using a recently proposed spin–generating method [8], we construct from these static solutions new rotating Einstein–Maxwell ring solutions with only a cosmic ring singularity. These solutions depend on four parameters, the values of which can be chosen such that the elementary particle constraints (1) (slightly generalized to $g \approx 2$) and (3) are satisfied. However, it then turns out that for the "elementary" orders of magnitude $|J| \sim Q^2 \sim m_P^2$, the mass of these objects cannot be small, but is also of the order of the Planck mass. We show that, in the case of large quantum numbers ($|J| \gg m_P^2$), a subclass of these solutions describes macroscopic charged or neutral rotating cosmic rings, also satisfying the elementary particle constraint (3), but with $g \neq 2$.

Under the assumption of a timelike Killing vector field ∂_t, the spacetime metric and electromagnetic field may be parametrized by

$$\begin{aligned} ds^2 &= f\,(dt - \omega_i dx^i)^2 - f^{-1}\,h_{ij}\,dx^i dx^j\,, \\ F_{i0} &= \partial_i v\,, \qquad F^{ij} = f\,h^{-1/2}\epsilon^{ijk}\partial_k u\,, \end{aligned} \tag{4}$$

where the various fields depend only on the three space coordinates x^i. The stationary Einstein–Maxwell equations are equivalent to the three–dimensional Ernst equations [9]

$$\begin{aligned} f\nabla^2\mathcal{E} &= \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f\nabla^2\psi &= \nabla\psi\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f^2 R_{ij}(h) &= Re\left[\frac{1}{2}\mathcal{E},_{(i}\overline{\mathcal{E}},_{j)} + 2\psi\mathcal{E},_{(i}\overline{\psi},_{j)} - 2\mathcal{E}\psi,_{(i}\overline{\psi},_{j)}\right]\,, \end{aligned} \tag{5}$$

where the scalar products and Laplacian are computed with the metric h_{ij}, the complex Ernst potentials $\mathcal{E}$ and ψ are defined by

$$\mathcal{E} = f - \overline{\psi}\psi + i\chi\,, \qquad \psi = v + iu\,, \tag{6}$$

and χ is the twist potential

$$\partial_i\chi = -f^2\,h^{-1/2}h_{ij}\,\epsilon^{jkl}\partial_k\omega_l + 2(u\partial_i v - v\partial_i u)\,. \tag{7}$$

These equations are invariant under an SU(2,1) group of transformations [10]. The class of electrostatic solutions ($\mathcal{E}$ and ψ real) depending on a single real potential can be reduced, by a group transformation, to $\mathcal{E} = \mathcal{E}_0$ constant, which solves the first equation (5). The form of the solution of the second equation (5) depends on the sign of $\mathcal{E}_0$. Representative solutions are

$$\begin{aligned} \mathcal{E}_0 &= -1\,, & \psi_0 &= \coth(\sigma)\,, & f_0 &= 1/\sinh^2\sigma \\ \mathcal{E}_0 &= 0\,, & \psi_0 &= 1/\sigma\,, & f_0 &= 1/\sigma^2 \\ \mathcal{E}_0 &= +1\,, & \psi_0 &= \cot(\sigma)\,, & f_0 &= 1/\sin^2\sigma \end{aligned} \tag{8}$$

where the potential $\sigma(\boldsymbol{x})$ is harmonic ($\nabla^2\sigma = 0$); other electrostatic solutions depending on a single potential may be obtained from these by SU(2,1) transformations. We note that the electric and gravitational potentials (8) are singular for $\sigma = 0$ if $\mathcal{E}_0 = -1$ or 0, and for $\sigma = n\pi$ (n integer) if $\mathcal{E}_0 = +1$.

As we wish to obtain axisymmetric ring–like solutions, we choose oblate spheroidal coordinates (x, y), related to the familiar Weyl coordinates (ρ, z) by

$$\begin{aligned} \rho &= \nu\,(1+x^2)^{1/2}(1-y^2)^{1/2}\,, \\ z &= \nu\,xy\,. \end{aligned} \tag{9}$$

In these coordinates, the three–dimensional metric $d\sigma^2 \equiv h_{ij}dx^i dx^j$

$$d\sigma^2 = \nu^2\,[\mathrm{e}^{2k}(x^2+y^2)(\frac{dx^2}{1+x^2} + \frac{dy^2}{1-y^2}) + (1+x^2)(1-y^2)\,d\varphi^2\,] \tag{10}$$

depends on the single function $k(x, y)$. Now, following Bronnikov et al. [7], we assume the harmonic potential σ to depend only on the variable x, which yields

$$\sigma = \sigma_0 + \alpha \arctan x\,, \qquad \mathrm{e}^{2k} = \left(\frac{1+x^2}{x^2+y^2}\right)^{\mathcal{E}_0\alpha^2}\,, \tag{11}$$

where σ_0 and α are integration constants. We note that the reflexion $x \leftrightarrow -x$ is an isometry for the three–dimensional metric (10), which has two points at infinity $x = \pm\infty$. The full four–dimensional metric (4) is quasi–regular (i.e. regular except on the ring $x = y = 0$, see below) for $x \in R$ if

$$\begin{array}{lll} |\sigma_0| > |\alpha|\pi/2 & & \text{for } \mathcal{E}_0 = -1 \text{ or } 0 \\ (n+|\alpha|/2)\pi < \sigma_0 < (n+1-|\alpha|/2)\pi & (|\alpha| < 1) & \text{for } \mathcal{E}_0 = +1 \end{array} \tag{12}$$

for some integer n. If these conditions are fulfilled, this metric describes a wormhole spacetime with two asymptotically flat regions connected through the disk $x = 0$ ($z = 0, \rho < \nu$). There is no horizon. The point singularity of the spherically symmetric (Reissner–Nordström) solution is here spread over the ring $x = y = 0$ ($z = 0, \rho = \nu$), near which the behavior of the spatial metric

$$d\sigma^2 \simeq \nu^2\,[(x^2+y^2)^{1-\mathcal{E}_0\alpha^2}(dx^2+dy^2) + d\varphi^2] \tag{13}$$

is that of a cosmic string with deficit angle $\pi(\mathcal{E}_0\alpha^2 - 1)$, which is negative in all cases of interest (it can be positive only for $\mathcal{E}_0 = +1$, $|\alpha| > 1$, corresponding to a singular solution). This ring singularity disappears in the limit of a vanishing deficit angle ($\mathcal{E}_0 = +1, |\alpha| \to 1$), where the solution reduces to a Reissner–Nordström solution with naked point singularity. The asymptotic behaviours of the gravitational and electric potentials at the two points at infinity are those of particles with masses and charges

$$M_\pm = \mp\alpha\nu\frac{\psi_0(\pm\infty)}{\sqrt{f_0(\pm\infty)}}\,, \qquad Q_\pm = \pm\alpha\nu\,; \tag{14}$$

the three cases (8) lead respectively to $Q_\pm^2 < M_\pm^2$ for $\mathcal{E}_0 = -1$, $Q_\pm^2 = M_\pm^2$ for $\mathcal{E}_0 = 0$, and $Q_\pm^2 > M_\pm^2$ for $\mathcal{E}_0 = +1$. The vanishing of the sum of the outgoing electric fluxes at $x = \pm\infty$ shows that the ring $x = y = 0$ is uncharged.

A case of special interest is $\mathcal{E}_0 = +1$, $\sigma_0 = \pi/2$, corresponding to a symmetrical wormhole metric [13]. The mass of this particle $M_\pm = \alpha\nu\sin(\alpha\pi/2)$ does not depend on the point at infinity considered, and is positive, even though the deficit angle is negative. For the physical characteristics of this particle to be those of a spinless electron, we should take $|\alpha| \sim m_e/m_P$, and $\nu \sim m_P^2/m_e$, of the order of the classical electron radius.

Now we proceed to generate asymptotically flat rotating solutions from these static axisymmetric solutions, using the simple procedure recently proposed in [8]. This procedure Σ (generalized in [11]) involves three successive transformations:

1) The electrostatic solution (real potentials $\mathcal{E}$, ψ, e^{2k}) is transformed to another electrostatic solution ($\hat{\mathcal{E}}$, $\hat{\psi}$, $\mathrm{e}^{2\hat{k}}$) by the SU(2,1) involution Π:

$$\hat{\mathcal{E}} = \frac{-1+\mathcal{E}+2\psi}{1-\mathcal{E}+2\psi}\,, \qquad \hat{\psi} = \frac{1+\mathcal{E}}{1-\mathcal{E}+2\psi}\,, \quad \mathrm{e}^{2\hat{k}} = \mathrm{e}^{2k}\,. \tag{15}$$

In the case of asymptotically flat fields with large distance monopole behavior, if the gauge is chosen so that $f(\infty) = (1+\psi(\infty))^2$, then the asymptotic behaviors of the resulting electric and gravitational potentials are those of the Bertotti–Robinson solution [12], $\hat{\psi} \propto r$, $\hat{f} \propto r^2$.

2) The static solution ($\hat{\mathcal{E}}$, $\hat{\psi}$, $\mathrm{e}^{2\hat{k}}$) is transformed to a uniformly rotating frame by the global coordinate transformation $d\varphi = d\varphi' + \Omega\,dt'$, $dt = dt'$, leading to gauge–transformed complex fields $\hat{\mathcal{E}}'$, $\hat{\psi}'$, $\mathrm{e}^{2\hat{k}'}$. While such a transformation on an asymptotically Minkowskian metric leads to a non–asymptotically Minkowskian metric, it does not modify the leading asymptotic behavior of the Bertotti–Robinson metric, so that the transformed fields are again asymptotically Bertotti–Robinson.

3) The solution ($\hat{\mathcal{E}}'$, $\hat{\psi}'$, $\mathrm{e}^{2\hat{k}'}$) is transformed back by the involution Π to a solution ($\mathcal{E}'$, ψ', $\mathrm{e}^{2k'}$) which, by construction, is asymptotically flat, but now has asymptotically dipole magnetic and gravimagnetic fields. As shown in [8], the combined transformation Σ transforms the Reissner–Nordström family of solutions into the Kerr–Newman family.

The static ring wormhole solutions of the preceding section have two distinct asymptotically flat regions $x \to \pm\infty$. To apply the general spin–generating procedure Σ to such wormhole spacetimes, we must therefore select a particular region at infinity. e. g. $x \to +\infty$, and gauge transform the static solution $(\mathcal{E}_0, \psi_0(x))$ to

$$\mathcal{E}(x) = c^2\mathcal{E}_0 - 2cd\psi_0(x) - d^2\,, \qquad \psi(x) = c\psi_0(x) + d \tag{16}$$

with the parameters $c^2 = 1/f_0(+\infty)$, $d = -c\psi_0(+\infty)$ such that $\psi(+\infty) = 0$, $f(+\infty) = 1$. A perturbative approach to the generation of slowly rotating ring wormhole solutions [13] shows that the asymmetry between the two points at infinity thus introduced is a necessary feature of spinning ring wormholes.

To recover the spacetime metric from the spinning potentials ($\mathcal{E}'$, ψ', $\mathrm{e}^{2k'}$), we compute from (7) the partial derivative $\partial_y\omega'_\varphi(x,y)$, which is a rational function of y, and integrate it with the boundary condition $\omega'_\varphi(x,\pm1) = 0$ (regularity on the axis $\rho = 0$). The resulting spacetime metric is of the form (4), (10) with the metric functions

$$\begin{aligned} f' &= |\Delta|^{-2}(1-\Omega^2\rho^2/\hat{f}^2)\,f\,, \qquad f'^{-1}\mathrm{e}^{2k'} = |\Delta|^2 f^{-1}\mathrm{e}^{2k}\,, \\ \omega'_\varphi &= \Omega\nu^2(1-y^2)\left[\frac{|\Delta_0|^2(1+x^2)}{b^2\psi^2(\hat{f}^2-\Omega^2\rho^2)} - \frac{\hat{f}^2\xi^2}{1+x^2} + \eta(\eta-\alpha\frac{b}{c})\right]\,, \end{aligned} \tag{17}$$

where $b = d+1$, $\hat{f}(x) = f(x)/b^2\psi^2(x)$, $\Delta_0(x) = \Delta(x, y_0(x))$ with $1-y_0^2 = \hat{f}^2/\Omega^2\nu^2(1+x^2)$, and

$$\begin{aligned} \Delta(x,y) &= 1 + \Omega^2\nu^2 b\psi(\alpha^2 b^2/2c^2 + \xi(1-y^2)) - i\Omega\nu b\eta\psi y\,, \\ \xi(x) &= (1+x^2)/2\hat{f} - \alpha^2b^2/2c^2\,, \quad \eta(x) = x + \alpha(2b-1)/c - \alpha/c\psi\,. \end{aligned} \tag{18}$$

We can show that zeroes of $\Delta(x,y)$, corresponding to strong Kerr–like ring singularities [11] of the stationary solution, are absent if

$$\alpha b c > 0\,. \tag{19}$$

This quasi–regularity condition which, in the cases $\mathcal{E}_0 = -1$ or 0, is equivalent to assuming the static mass M_+ to be positive, can always be satisfied by choosing the appropriate sign for the constant c in (16). The metric (17) is of course still singular on the rotating cosmic ring $x = y = 0$, with the same deficit angle $\pi(\mathcal{E}_0\alpha^2 - 1)$ as in the static case. The gravitational potential f' vanishes on the stationary limit surfaces $\hat{f}(x) = \pm\Omega\rho$, where the full metric is regular. However the spinning solution (17) is horizonless, just as the corresponding static solution [11]. This spinning solution is by construction asymptotically flat for $x \to +\infty$, but not for $x \to -\infty$, where the asymptotic metric behaves as

$$ds'^2 \simeq -l^{-2}\rho^{-2}(dt + (\Omega/4)(\rho^2 + 4z^2)\,d\varphi)^2 - 16\Omega^{-2}l^6\rho^4(d\rho^2 + dz^2) + l^4\rho^4 d\varphi^2 \tag{20}$$

(with $l^2 = \Omega^2/4\hat{f}(-\infty)$). The study of geodesic motion in the metric (17) shows that all test particles coming from $x \to +\infty$ are reflected back by an infinite potential barrier at the stationary limit, so that there is no loss of information to $x \to -\infty$.

From the asymptotic behaviours of the spinning metric and electromagnetic field, we obtain the corresponding mass, angular momentum, charge, and magnetic dipole moment,

$$\begin{aligned} M &= (\nu\alpha/c)(b - 1 + \tau)\,, \qquad & J &= \nu\beta(M + \delta)\,, \\ Q &= (\nu\alpha/c)(1 - \tau)\,, \qquad & \mu &= \nu\beta(Q - \delta)\,, \end{aligned} \tag{21}$$

with $\beta = \Omega\nu\alpha^2b^2/c^2$, $\tau = \beta^2c^2/2\alpha^2 b$, $\delta = \nu c(1 - \mathcal{E}_0\alpha^2)/3\alpha b$.

Can the values of these parameters correspond to those of elementary particles? Combining the above values we obtain

$$M^2 - Q^2 - a^2 = \nu^2(\beta^2 - \mathcal{E}_0\alpha^2) - a^2\,. \tag{22}$$

The quasi–regularity condition (19) implies $\delta > 0$, so that $a^2 > \nu^2\beta^2$, hence the inequality (3) is satisfied for $\mathcal{E}_0 \geq 0$. The gyromagnetic ratio

$$g = 2\frac{M(Q-\delta)}{Q(M+\delta)} \tag{23}$$

can be very close to 2 for very small values of δ. One would then expect that the values of the independent free parameters ν, α, β, and τ can be adjusted so that the four physical parameters (21) take their elementary particle values. However, owing to the regularity constraint (12) which restricts the range of allowed parameter values, it turns out that if for instance the charge and angular momentum are of order unity (in Planck units) and

$g \simeq 2$, then the mass of these spinning ring "particles" should be at least of the order of the Planck mass.

In the case of large quantum numbers ($|J| \gg m_P^2$), our classical solutions (17) describe macroscopic closed cosmic strings with negative deficit angle, but positive total mass. These cosmic strings satisfy the elementary particle constraint (3) if $\mathcal{E}_0 \geq 0$. The exotic line source $x = y = 0$ is spacelike only if it lies outside the stationary limit surface, which further restricts the parameter values. A specially interesting case is $\tau = 1$, corresponding to a neutral spinning cosmic string ($Q = 0$). In this case we can show that the ring source is spacelike for $|\beta| < 1/2$ (leading to $\delta/M > 4/3$); for $\mathcal{E}_0 = 0$, the range of $|\beta|$ can be narrowed down to $0.29 < |\beta| < 0.40$. These neutral strings have a proper perimeter of the order of or smaller than their Schwarzschild radius $M = |\beta|\nu$, a rotation velocity which is close to 1, and a magnetic moment $|\mu| \geq \nu^2/3$, corresponding to a current intensity of the order of the Planck intensity. The stability of these exact closed string solutions remains to be investigated.

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